Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/02 Pure Mathematics and Statistics
Question Paper

Wednesday 13 June 2018 – Morning
Time allowed: 2 hours

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

Model Answers

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **12** pages.

1 Show that $\sqrt{27}+\sqrt{192}=a\sqrt{b}$, where a and b are prime numbers to be determined. **[2]**

$\sqrt{27}+\sqrt{192} = \sqrt{9\times3}+\sqrt{64\times3} = 3\sqrt{3}+8\sqrt{3}$

$= 11\sqrt{3}$

$a=11$, $b=3$

2 Solve the inequality $|2x+1|<5$. **[3]**

$|2x+1|<5$

$-5<2x+1<5$

$-6<2x<4$

$-3<x<2$

3 The probability that Chipping FC win a league football match is $P(W) = 0.4$.

(i) Calculate the probability that Chipping FC fail to win each of their next two league football matches. **[1]**

$P(W^c) = 1-0.4 = 0.6$

$P(W^c)P(W^c) = 0.6^2 = 0.36$

The probability that Chipping FC lose a league football match is $P(L) = 0.3$.

(ii) Explain why $P(W)+P(L)\neq 1$. **[1]**

They could draw, so winning and losing are not the only two possibilities

4 A survey of the number of cars per household in a certain village generated the data in Fig. 4.

Number of cars	0	1	2	3	4
Number of households	8	22	31	27	7

Fig. 4

(i) Calculate the mean number of cars per household. **[1]**

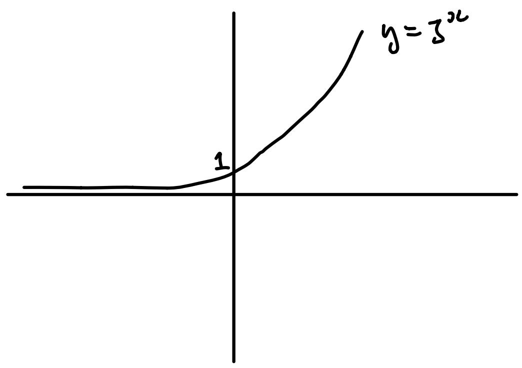
$$\text{Mean} = \frac{8(0)+22(1)+31(2)+27(3)+7(4)}{8+22+31+27+7} = \frac{193}{95}$$

$$= 2.03$$

(ii) Calculate the standard deviation of the number of cars per household. **[1]**

1.076

5 **(i)** (*A*) Sketch the graph of $y = 3^x$. **[1]**



(*B*) Give the coordinates of any intercepts. **[1]**

$(0, 1)$

The curve $y = \mathrm{f}(x)$ is the reflection of the curve $y = 3^x$ in the line $y = x$.

(ii) Find $\mathrm{f}(x)$. **[1]**

The reflection in $y=x$ is the inverse of $f(x)=3^x$

$\Rightarrow y = f(x) = \log_3(x)$

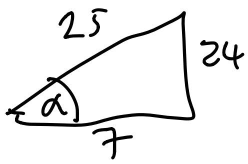
6 **(i)** Express $7\cos x - 24\sin x$ in the form $R\cos(x+\alpha)$, where $0 < \alpha < \frac{\pi}{2}$. **[3]**

$7\cos(x) - 24\sin(x) = R\cos(x+\alpha)$

$= R(\cos(x)\cos(\alpha) - \sin(x)\sin(\alpha))$

$\Rightarrow 7 = R\cos(\alpha)\ ,\quad 24 = R\sin(\alpha)$

$\frac{R\sin(\alpha)}{R\cos(\alpha)} = \frac{24}{7} = \tan(\alpha)\ ,\quad \alpha = 1.287$



$\cos(\alpha) = \frac{7}{25}$

$7 = R\cos(\alpha) = R \cdot \frac{7}{25}$

$\Rightarrow R = 25$

$\therefore\ 7\cos(x) - 24\sin(x) = 25\cos(x + 1.287)$

(ii) Write down the range of the function

$$f(x) = 12 + 7\cos x - 24\sin x\ , \qquad 0 \leqslant x \leqslant 2\pi .$$

$f(x) = 12 + 7\cos(x) - 24\sin(x)$

$= 12 + 25\cos(x + 1.287)$

$12 - 25 \leq f(x) \leq 12 + 25$

$-13 \leq f(x) \leq 37$

7 Find $\int\left(4\sqrt{x} - \frac{6}{x^3}\right)dx$. **[4]**

$\int\left(4\sqrt{x} - \frac{6}{x^3}\right)dx = \int 4x^{\frac{1}{2}} - 6x^{-3}\,dx = \frac{8}{3}x^{\frac{3}{2}} + 3x^{-2} + c$

Section B (79 marks)

8 Every morning before breakfast Laura and Mike play a game of chess. The probability that Laura wins is 0.7. The outcome of any particular game is independent of the outcome of other games. Calculate the probability that, in the next 20 games,

$X \sim B(20, 0.7)$

(i) Laura wins exactly 14 games, **[2]**

$P(X=14) = \binom{20}{14}(0.7)^{14}(0.3)^{6} = 0.1916$

(ii) Laura wins at least 14 games. **[2]**

$P(X \leq 13) = 0.608$

9 At the end of each school term at North End College all the science classes in year 10 are given a test. The marks out of 100 achieved by members of set 1 are shown in Fig. 9.

3	5
4	0 9
5	2 3 6
6	0 1 3 5 6
7	0 1 2 5 6 8 9 9
8	3 4 6 6 8 8 9
9	5 5 5 6 7

Key 5 | 2 represents a mark of 52

Fig. 9

(i) Describe the shape of the distribution. **[1]**

Negative Skew

(ii) The teacher for set 1 claimed that a typical student in his class achieved a mark of 95. How did he justify this statement? **[1]**

He used the mode

(iii) Another teacher said that the average mark in set 1 is 76. How did she justify this statement? **[1]**

She used the median

Benson's mark in the test is 35. If the mark achieved by any student is an outlier in the lower tail of the distribution, the student is moved down to set 2.

(iv) Determine whether Benson is moved down to set 2. **[2]**

$61 - 1.5(88-61) = 20.5$

$Q_1 = 61$ $Q_3 = 88$

$35 > 20.5$ so 35 is not an outlier.

He does not need to move down to set 2.

10 The screenshot in Fig. 10 shows the probability distribution for the continuous random variable X, where $X \sim N(\mu, \sigma^2)$.

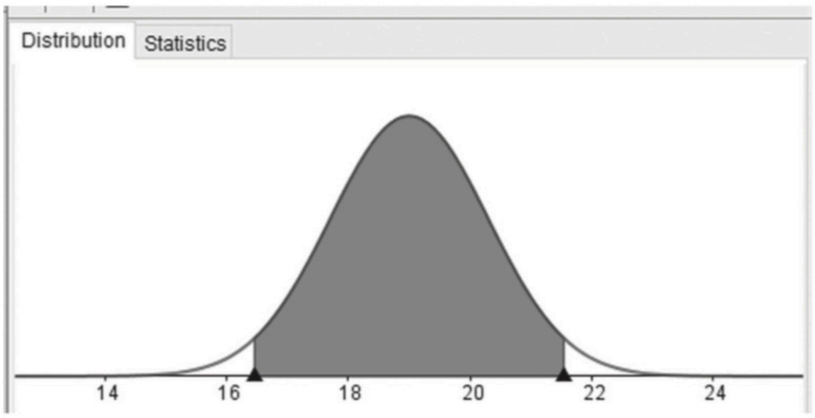


Fig. 10

The area of each of the unshaded regions under the curve is 0.025. The lower boundary of the shaded region is at 16.452 and the upper boundary of the shaded region is at 21.548.

(i) Calculate the value of μ. **[1]**

$$\frac{21.548 + 16.452}{2} = 19$$

(ii) Calculate the value of σ^2. **[3]**

$1 - 0.025 - 0.025 = 0.95$

95% of the data is within the shaded area. The critical value for 95% is 1.96:

$$1.96 = \frac{21.548 - 19}{\sigma} \Rightarrow \sigma = 1.3 \text{ so } \sigma^2 = 1.69$$

(iii) Y is the random variable given by $Y = 4X + 5$.

(A) Write down the distribution of Y. **[3]**

$E(Y) = 4E(X) + 5 = 4 \times 19 + 5 = 81$

$Var(Y) = 4^2 Var(X)$

$= 16 \cdot 1.69$

$= 27.04$

$= 5.2^2$

$\therefore Y \sim N(81, 5.2^2)$

(B) Find P(Y > 90). **[1]**

$P(Y > 90) = P\left(\frac{Y - 81}{5.2} > \frac{90 - 81}{5.2}\right)$

$= P(Z > 1.73)$

$= 1 - 0.9582$

$= 0.0418$

11 The discrete random variable X takes the values 0, 1, 2, 3, 4 and 5 with probabilities given by the formula

$$P(X = x) = k(x + 1)(6 - x).$$

(i) Find the value of k. **[2]**

Total probability $= 1$

$\Rightarrow \quad k(1)(6) + k(2)(5) + k(3)(4) + k(4)(3) + k(5)(2) + k(6)(1) = 1$

$k(6 + 10 + 12 + 12 + 10 + 6) = 1$

$56k = 1$

$k = 1/56$

In one half-term Ben attends school on 40 days. The probability distribution above is used to model X, the number of lessons per day in which Ben receives a gold star for excellent work.

(ii) Find the probability that Ben receives no gold stars on each of the first 3 days of the half-term and two gold stars on each of the next 2 days. **[2]**

$= p(X=0)^3 \cdot p(X=2)^2$

$= \left[\frac{1}{56}(1)(6)\right]^3 \cdot \left[\frac{1}{56}(3)(4)\right]^2$

$= 0.00005648$

(iii) Find the expected number of days in the half-term on which Ben receives no gold stars. **[2]**

$= 40 \times \frac{6}{56} = 4.286$

12 **You must show detailed reasoning in this question.**

In the summer of 2017 in England a large number of candidates sat GCSE examinations in **both** mathematics **and** English. 56% of these candidates achieved at least level 4 in mathematics and 80% of these candidates achieved at least level 4 in English. 14% of these candidates did not achieve at least level 4 in either mathematics or English.

Determine whether achieving level 4 or above in English and achieving level 4 or above in mathematics were independent events. **[5]**

Let M = event of achieving level 4 or above in maths

Let E = event of achieving level 4 or above in English

$p(M) = 0.56$, $p(E) = 0.8$

$p(M^c \cup E^c) = 0.14$, $p(M \cup C) = 1 - p(M^c \cup E^c)$

$= 1 - 0.14$

$= 0.86$

$p(M \cap E) = p(E) + p(M) - p(M \cup E)$

$= 0.8 + 0.56 - 0.86 = 0.5$

$p(M)p(E) = 0.56 \times 0.8 = 0.448$.

$0.448 \neq 0.5$ so the events are not independent.

13 Each weekday Keira drives to work with her son Kaito. She always sets off at 8.00 a.m. She models her journey time, x minutes, by the distribution $X \sim \mathrm{N}(15, 4)$.

Over a long period of time she notes that her journey takes less than 14 minutes on 7% of the journeys, and takes more than 18 minutes on 31% of the journeys.

(i) Investigate whether Keira's model is a good fit for the data. **[3]**

$X \sim N(15,4)$

$p(X<14) = p\left(\frac{X-15}{2} < \frac{14-15}{2}\right)$

$= p(Z < -\frac{1}{2})$

$= 1 - 0.6915$

$= 0.3085$

$p(X>18) = p(Z > \frac{3}{2})$

$= 1 - 0.9332$

$= 0.0668$

0.3085 is not close to 7% (0.07) and 0.0668 is not close to 31% (0.31) so the figures do not support the model.

Kaito believes that Keira's value for the variance is correct, but realises that the mean is not correct.

(ii) Find, correct to two significant figures, the value of the mean that Keira should use in a refined model

$p(X<14) = 0.07$

$p\left(Z < \frac{14-\mu}{2}\right) = 0.07$

$\frac{14-\mu}{2} = -1.48$

$\mu = 14 + 2.96$

$= 16.96$

Keira buys a new car. After driving to work in it each day for several weeks, she randomly selects the journey times for n of these days. Her mean journey time for these n days is 16 minutes. Using the refined model she conducts a hypothesis test to see if her mean journey time has changed, and finds that the result is significant at the 5% level.

(iii) Determine the smallest possible value of n. **[5]**

The critical value for 5% is 1.96.

$$\frac{16-\mu}{\frac{2}{\sqrt{n}}} > 1.96$$

$$16-15 > 1.96\left(\frac{2}{\sqrt{n}}\right)$$

$$\sqrt{n} > 2\times 1.96$$

$$\sqrt{n} > 3.92$$

$$n > 15.37$$

$$n = 16$$

14 The pre-release material includes data on unemployment rates in different countries. A sample from this material has been taken. All the countries in the sample are in Europe. The data have been grouped and are shown in Fig 14.1.

Unemployment rate	0–	5–	10–	15–	20–	35–50
Frequency	15	21	5	5	2	2

Fig. 14.1

A cumulative frequency curve has been generated for the sample data using a spreadsheet. This is shown in Fig. 14.2.

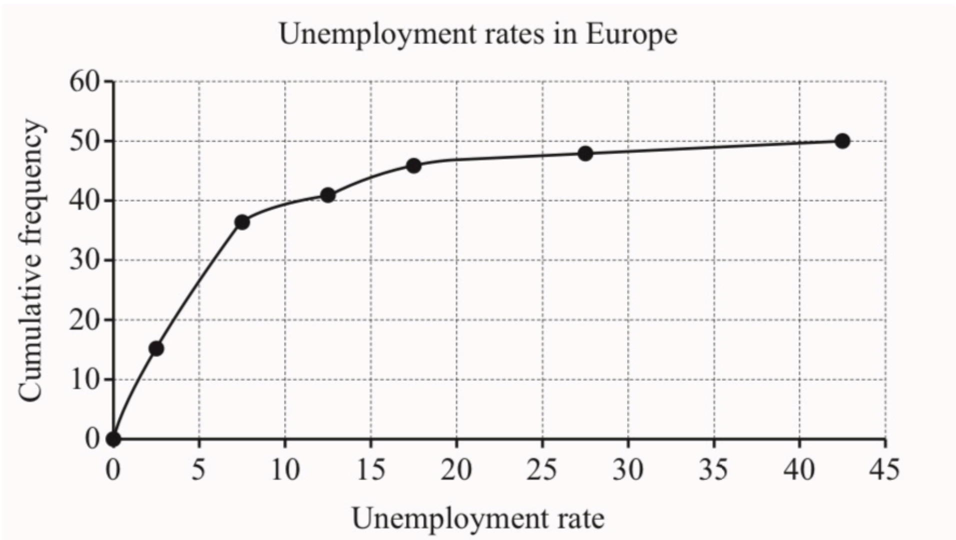


Fig. 14.2

Hodge used Fig. 14.2 to estimate the median unemployment rate in Europe. He obtained the answer 5.0. The correct value for this sample is 6.9.

(i) (*A*) There is a systematic error in the diagram.

- Identify this error.
- State how this error affects Hodge's estimate. **[2]**

(*B*) There is another factor which has affected Hodge's estimate.

- Identify this factor.
- State how this factor affects Hodge's estimate. **[2]**

A) The cumulative frequencies have been plotted against the midpoints of the class intervals which reduces the estimate.

B) He has used the grouped data from the graph rather than the raw data which reduced the error introduced by plotting.

(ii) Use your knowledge of the pre-release material to give another reason why any estimation of the median unemployment rate in Europe may be unreliable. **[1]**

Exact unemployment percentage is often estimated.

(iii) Use your knowledge of the pre-release material to explain why it is very unlikely that the sample has been randomly selected from the pre-release material. **[1]**

The pre-release material contains countries from other continents rather than Europe. It is unlikely that a random sample would only include European countries.

The scatter diagram shown in Fig. 14.3 shows the unemployment rate and life expectancy at birth for the 47 countries in the sample for which this information is available.

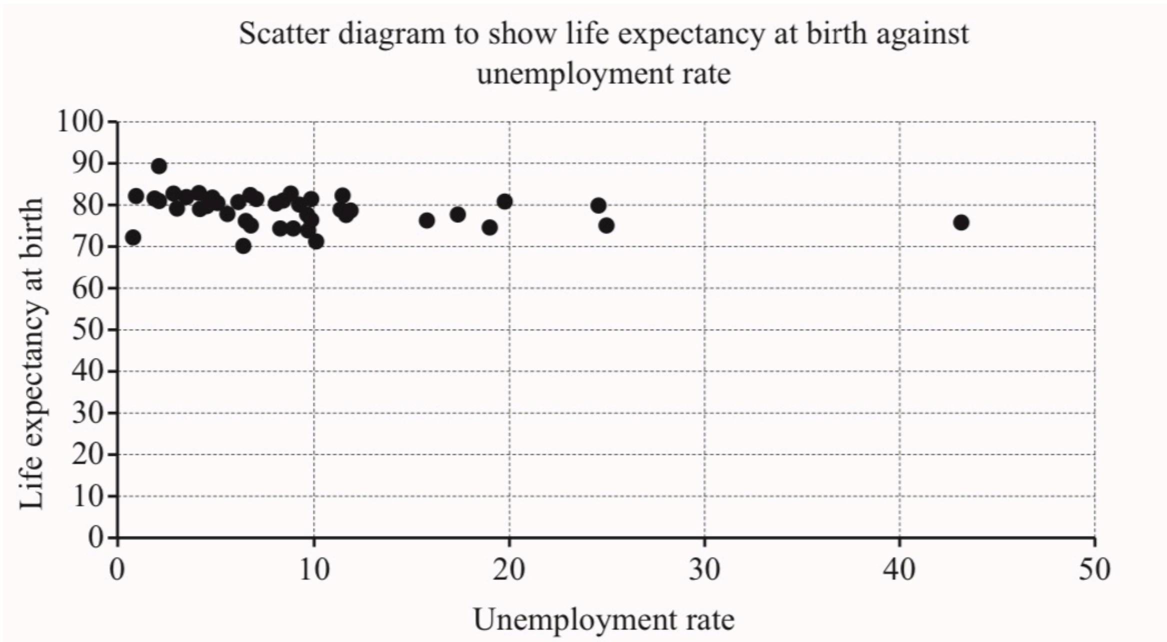


Fig. 14.3

The product moment correlation coefficient for the 47 items in the sample is – 0.2607.
The p-value associated with $r = -0.2607$ and $n = 47$ is 0.0383.

(iv) Does this information suggest that there is an association between unemployment rate and life expectancy at birth in countries in Europe? **[2]**

The value of r suggests a negative correlation. $0.0383 < 0.05$ so the result is significant at 5%. $0.0383 > 0.01$ so it is not significant at 1%.

(v) The unemployment rate in Kosovo is 35.3, but there is no data available on life expectancy. Is it reasonable to use Hodge's line of best fit to estimate life expectancy at birth in Kosovo? **[1]**

The use of a line of best fit is probably not appropriate because of the weak correlation. Therefore we should not use this to estimate the life expectancy in Kosovo.

15 You must show detailed reasoning in this question.

The equation of a curve is

$$y^3 - xy + 4\sqrt{x} = 4 .$$

Find the gradient of the curve at each of the points where $y = 1$. **[9]**

Implicit differentiation:

$$3y^2\frac{dy}{dx} - y - x\frac{dy}{dx} + 2x^{-\frac{1}{2}} = 0$$

when $y=1$: $y^3 - xy + 4\sqrt{x} = 4$

$$1 - x + 4\sqrt{x} = 4$$

$$x - 4\sqrt{x} + 3 = 0$$

$$(\sqrt{x}-3)(\sqrt{x}-1)$$

$$\sqrt{x} = 3 \quad \text{or} \quad \sqrt{x} = 1$$

$$x = 9 \quad \text{or} \quad x = 1$$

Sub into the differentiated equation:

$(9,1)$: $3(1)^2\frac{dy}{dx} - (1) - 9\frac{dy}{dx} + \frac{2}{\sqrt{9}} = 0$

$$-6\frac{dy}{dx} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\frac{dy}{dx} = -\frac{1}{18}$$

$(1,1)$: $3\frac{dy}{dx} - 1 - \frac{dy}{dx} + 2 = 0$

$$2\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

So the two gradients are $-\frac{1}{18}$ and $-\frac{1}{2}$.

16 In the first year of a course, an A-level student, Aaishah, has a mathematics test each week. The night before each test she revises for t hours. Over the course of the year she realises that her percentage mark for a test, p, may be modelled by the following formula, where A, B and C are constants.

$$p = A - B(t - C)^2$$

- Aaishah finds that, however much she revises, her maximum mark is achieved when she does 2 hours revision. This maximum mark is 62.
- Aaishah had a mark of 22 when she didn't spend any time revising.

(i) Find the values of A, B and C. **[3]**

The max of p is at A which happens when $t-C=0$.

So $max = A = 62$.

She achieved this when $t=2$:

$t - c = 0$
$2 - c = 0$
$c = 2$

When $t=0$, $p = A - B(0-c)^2$

$22 = 62 - B(2^2)$

$-40 = -4B$

$B = 10$

$\therefore p = 62 - 10(t-2)^2$

(ii) According to the model, if Aaishah revises for 45 minutes on the night before the test, what mark will she achieve? **[2]**

$t = 45 = 0.75$ hours

$p = 62 - 10(0.75-2)^2$
$= 62 - 10(1.5675)$
$= 46.375$

She will get 46 marks.

(iii) What is the maximum amount of time that Aaishah could have spent revising for the model to work? **[2]**

$$62-10((\ 2)\ -0$$

$$62 \geq 10(t-2)^2$$

$$6.2 \geq (t-2)^2$$

$$2.49 \geq t-2$$

$$t \leq 4.49$$

So the maximum time she could have spent revising is 4 hours 30 minutes.

In an attempt to improve her marks Aaishah now works through problems for a total of t hours over the three nights before the test. After taking a number of tests, she proposes the following new formula for p.

$$p = 22 + 68(1 - e^{-0.8t})$$

For the next three tests she recorded the data in Fig. 16.

t	1	3	5
p	59	84	89

Fig. 16

(iv) Verify that the data is consistent with the new formula. **[2]**

When $t=1$, $p = 22+68(1-e^{-0.8})$

$= 59.45$

≈ 59

$t=3$, $p = 22+68(1-e^{-2.4})$

$= 83.83$

≈ 84

$t=5$, $p = 22+68(1-e^{-4})$

$= 88.75$

≈ 89

The data is consistent with the new formula as the values obtained are approximately the same as the values in Fig.16.

(v) Aaishah's tutor advises her to spend a minimum of twelve hours working through problems in future. Determine whether or not this is good advice. **[2]**

For $t=12$, $p = 22+68(1-e^{-9\times 0.6})$

$= 89.99658812$

$\simeq 90$

This is not good advice because Aaishah obtains 89 marks for working 5 hours and this would be working 7 hours extra for 1 mark.

17 **(i)** Express $\dfrac{(x^2-8x+9)}{(x+1)(x-2)^2}$ in partial fractions. **[5]**

$= \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

$x^2-8x+9 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

When $x=2$:

$4-16+9 = A(0)+B(3)(0)+C(3)$

$-3 = 3C$

$\Rightarrow C=-1$

When $x=-1$:

$1+8+9 = A(-3)^2+B(0)(-3)+C(0)$

$18 = 9A$

$\Rightarrow A=2$

When $x=0$

$9 = 4A-2B+C$

$9 = 8-2B-1$

$2=-2B$

$\Rightarrow B=-1$

So $\dfrac{(x^2-8x+9)}{(x+1)(x-2)^2} = \dfrac{2}{x+1} - \dfrac{1}{x-2} - \dfrac{1}{(x-2)^2}$

(ii) Express y in terms of x given that

$$\frac{dy}{dx} = \frac{y(x^2 - 8x + 9)}{(x+1)(x-2)^2} \text{ and } y = 16 \text{ when } x = 3.$$

[7]

$$\frac{dy}{dx} = \frac{y(x^2-8x+9)}{(x+1)(x-2)^2}$$

$$\int \frac{dy}{y} = \int \frac{x^2-8x+9}{(x+1)(x-2)^2}\,dx$$

$$\ln|y| = \int \frac{2}{x+1} - \frac{1}{x-2} \frac{-1}{(x-2)^2}\,dx$$

$$= 2\ln|x+1| - \ln|x-2| + \frac{1}{x-2} + C$$

Sub in $y=16$ and $x=3$

$$\ln(16) = 2\ln(4) - \ln(1) + 1 + C$$

$$= \ln(16) + 1 + C$$

$$\Rightarrow C = -1.$$

$$\ln|y| = 2\ln|x+1| - \ln|x-2| + \frac{1}{x-2} - 1$$

$$y = e^{[2\ln|x+1| - \ln|x-2| + \frac{1}{x-2} - 1]}$$

$$= e^{\ln(x+1)^2} \times e^{-\ln(x-2)} \times e^{\frac{1}{x-2}-1}$$

$$= \frac{(x+1)^2}{x-2} \times e^{\frac{1}{x-2}-x+2}$$

$$= \frac{(x+1)^2}{x-2} e^{\frac{3-x}{x-2}}$$

So $y = \frac{(x+1)^2}{x-2} e^{\frac{3-x}{x-2}}$